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Resolution of holomorphic functions by distributions.

Serre [5] proves the following duality theorem:

Let X be a compact complex manifold, $\dim X = n$, and let W be a holomorphic vector bundle on X, and W^* the dual bundle of W. Then the vector spaces $H^{p,q}(X,W) = H^q(X,\mathcal{O}^p(W))$ and

$H^{n-p,n-q}(X,W^*) = H^{n-q}(X,\mathcal{O}^{n-p}(W^*))$ are (canonically) dual to each other, in particular, they have the same (finite) dimension.

To prove this, he resolves the sheaf $\mathcal{O}^p(W)$ of germs of holomorphic p-forms with coefficients in a holomorphic vector bundle W in two (fine) ways:

$$0 \to \mathcal{O}^p(W) \to \mathcal{A}^{(p,o)}(W) \xrightarrow{\bar\partial} \mathcal{A}^{(p,1)}(W) \xrightarrow{\bar\partial} \dots \to 0$$

$$0 \to \mathcal{O}^p(W) \to \mathcal{D}'^{(p,o)}(W) \xrightarrow{\bar\partial} \mathcal{D}'^{(p,1)}(W) \xrightarrow{\bar\partial} \dots \to 0 ,$$

where $\mathcal{A}^{(p,q)}(W)$ is the sheaf of germs of $\mathcal{E}^\infty$- forms of type (p,q) with coefficients in W, and $\mathcal{D}'^{(p,q)}(W)$ is the same kind of distributional forms.

Thus one can calculate $H^q(X,\mathcal{O}^p(W))$ from either sequence. Since $\mathcal{D}'$ is dual to $\mathcal{A} = \mathcal{E}^\infty$, this is a natural procedure. The above result is a consequence of the well-known Grothendieck lemma, and of the fact that if $T \in \mathcal{D}'(U)$, U open in $\mathbb{C}^n$, satisfies $\partial T/\partial \bar{z}j = 0$ for $j = 1, \dots ,n$, then T is a holomorphic function. Concerning the Grothendieck lemma for distributions, Serre refers to a paper of Dolbeault [1], in which this distributional Grothendieck lemma is stated. However in this paper Dolbeault gives no proof, but says that this is an unpublished result of Grothendieck.

We propose to give a proof of the distributional Grothendieck lemma below. The proof is modelled on the method in Narasimhan [4]. According to Narasimhan, this is the method of Grothendieck, as exposed by Serre. We also prove the statement about the distribution T above, and this proof is a generalization of the 1 - dimensional proof in Gunning [2].
We first need the following lemma.

Let L', K, L be compact subsets of $\mathbb{R}^s$, $\mathbb{C}$, $\mathbb{C}^n$ respectively. Denote by (t, z, w) points in $\mathbb{R}^s \times \mathbb{C} \times \mathbb{C}^n$. Let $g \in \mathcal{D}'(U)$, where U is an open subset of $\mathbb{R}^s \times \mathbb{C} \times \mathbb{C}^n$ containing $L' \times K \times L$, and suppose $\partial g / \partial \bar{w}^k = 0$ for $1 \leq k \leq n$, where $w = (w', \ldots, w^n)$. Then there is a distribution $f \in$ '(U'), where U' is some open set contained in U and containing $L' \times K \times L$, such that $\partial f / \partial \bar{w}^k = 0$ for $1 \leq k \leq n$ and $\partial f / \partial \bar{z} = g$ in U'.

Proof: For simplicity, we assume that $s = 0$, as this does not affect the proof. We may also suppose that $g \in \mathcal{E}'(\mathbb{C} \times \mathbb{C}^n)$, since we can multiply g by a C_c^∞ function having support in U and being equal to 1 in a nbh. of $K \times L$, and then consider g in this last nbh.. Thus we assume g is a compactly supported distribution in $\mathbb{C} \times \mathbb{C}^n$. The proof is somewhat technical, and we divide it into three parts:

I) A statement needed fo define a distribution f in II).
II) Define f and prove that f is a distribution.
III) Show that f is the distribution we seek.

I) For $\varphi \in \mathcal{E}^{\infty}(\mathbb{C} \times \mathbb{C}^n)$, let $\tau_\alpha\varphi \in \mathcal{E}^{\infty}(\mathbb{C} \times \mathbb{C}^n)$ be the translation by $\partial \in \mathbb{C} \times \mathbb{C}^n$, thus $(\tau_\alpha\varphi)(z,w) := \varphi((z,w)-x)$.

If $\varphi \in \mathcal{E}_c^{\infty}(\mathbb{C} \times \mathbb{C}^n)$ and $\xi \in \mathbb{C}$, put $h(\xi) := \langle g, \tau_{(\xi,o)}\varphi\rangle$. Then $h \in C_c^{\infty}(\mathbb{C})$, and we have $\dfrac{\partial h}{\partial \bar\xi}(\xi) = -\langle g, \tau_{(\xi,o)} \dfrac{\partial \varphi}{\partial \bar z}\rangle$. (1)

We check this: If $0 \neq t \in \mathbb{R}$, then $\dfrac{h(\xi+t)-h(\xi)}{t} = \langle g, \dfrac{\tau_{(\xi+t,o)}\varphi - \tau_{(\xi,o)}\varphi}{t}\rangle$. Now $\lim\limits_{t\to o} \dfrac{\tau_{(\xi+t,o)}\varphi)(z,w) - (\tau_{(\xi,o)}\varphi)(z,w)}{t} =$

$$\lim_{t\to o} \frac{\varphi(z-\xi-t,w) - \varphi(z-\xi,w)}{t} = -\frac{\partial\varphi}{\partial\chi}(z-\xi,w) = -(\tau_{(\xi,o)}\frac{\partial\varphi}{\partial\chi})(z,w),$$

where $Z = x+iy$, with $\chi, y \in \mathbb{R}$. Similarly, if $0 \neq t \in \mathbb{R}$, then $\lim\limits_{t\to o} \dfrac{(\tau_{(\xi+it,o)}\varphi)(Z,w) - (\tau_{(\xi,o)}\varphi)(z,w)}{t}$

$$= -(\tau_{(\xi,o)}\frac{\partial\varphi}{\partial\varphi})(z,w).$$

We now prove that $\lim\limits_{t\to o} \dfrac{\tau_{(\xi+t,o)}\varphi - \tau_{(\xi,o)}\varphi}{t} = -\tau_{(\xi,o)}\dfrac{\partial\varphi}{\partial\chi}$ and $\lim\limits_{t\to o} \dfrac{\tau_{(\xi+it,o)}\varphi - \tau_{(\xi,o)}\varphi}{t} = -\tau_{(\xi,o)}\dfrac{\partial\varphi}{\partial y}$, for $t \in \mathbb{R}$, that is that these limits hold in the Fréchet space $C^{\infty}(\mathbb{C} \times \mathbb{C}^n)$. (2)

Consider the first limit. Let m be a non-negative integer and A a compact subset of $\mathbb{C} \times \mathbb{C}^n$. Denote for the moment by D

a differentiation monomial in the real coordinates of $\mathbb{C} \times \mathbb{C}^n$, and by $\| \|_{A,m}$ the semi-norm in $C^\infty(\mathbb{C} \times \mathbb{C}^n)$ given by A and m, using monomials of order $\leq m$. Then $\|\frac{1}{t}(\tau_{(\xi+t,0)}\varphi - \tau_{(\xi,0)}\varphi) + \tau_{(\xi,0)}\partial\varphi/\partial\chi\|_{A,m}$

$$= \sup_{\text{ord } D \leq m} \sup_{(z,w)\in A} | D(\tfrac{1}{t}(\tau_{(\xi+t,0)}\varphi - \tau_{(\xi,0)}\varphi) + \tau_{(\xi,0)}\partial\varphi/\partial\chi)(z,w) |.$$

Consider for instance the case $D = \partial/\partial u^k$, where $w^k = u^k + i v^k$ with $u^k, v^k \in \mathbb{R}$. Then $\frac{\partial}{\partial u^k}(\frac{1}{t}(\tau_{(\xi+t,0)}\varphi) - \tau_{(\xi,0)}\varphi) + \tau_{(\xi,0)}\partial\varphi/\partial\chi)(z,w)$

$$= (\tfrac{1}{t}(\tau_{(\xi+t,0)}\partial\varphi/\partial u^k - \tau_{(\xi,0)}\partial\varphi/\partial u^k) + \tau_{(\xi,0)}\partial^2\varphi/\partial\chi\partial u^k)(z,w).$$

Similar expressions hold for $\partial/\partial v^k$, $\partial/\partial\chi$, $\partial/\partial y$, and other monomials D. All these expressions tend uniformly to zero on the compact set A as t tends to zero. Hence the first limit expression in (2) above, and similarly also the second, is true. By the above expressions we therefore get, since g is contiuous on $C^\infty(\mathbb{C} \times \mathbb{C}^n)$, that $\partial h/\partial\xi_1$, and $\partial h/\partial\xi_2$ exist, where $\xi = \xi_1 + i\xi_2$ with $\xi_1, \xi_2 \in \mathbb{R}$, and further

$$\frac{\partial h}{\partial\xi_1}(\xi) = -\langle g, \tau_{(\xi,0)}\frac{\partial\varphi}{\partial\chi}\rangle \text{ and } \frac{\partial h}{\partial\xi_2}(\xi) = -\langle g, \tau_{(\xi,0)}\frac{\partial\varphi}{\partial y}\rangle.$$

This gives $\frac{\partial h}{\partial\bar\xi}(\xi) = -\langle g, \tau_{(\xi,0)}\frac{\partial\varphi}{\partial\bar z}\rangle$, and also, when applied several times, $\frac{\partial^{\alpha+\beta}h}{\partial\xi_1^\alpha\partial\xi_2^\beta}(\xi) = (-1)^{\alpha+\beta}\langle g, \tau_{(\xi,0)}\frac{\partial^{\alpha+\beta}\varphi}{\partial\chi^\alpha\partial y^\beta}\rangle$. The last shows that h is C^∞. We must show that h has compact support. Choose $R > 0$ and a compact set B in $\mathbb{C}^n$ such that $\text{supp } g \cup \text{supp } \varphi \subset K_R \times B$, where $K_R = \{|z| \leq R\}$.

If $|\xi| > 2R$ and $(z,w) \in K_R \times B$, then $|\xi - z| \geq ||\xi| - |z|| = |\xi| - |z| > R$, thus $(z-\xi, w) \notin K_R \times B$. Hence $\text{supp } g \cap \text{supp } \tau_{(\xi,0)}\varphi = \emptyset$

for $|\xi| > 2R$, and thus also $<g, \tau_{(\xi,0)}\varphi> = 0$ for $|\xi|>2R$, which gives supp $h \subset K_{2R}$.

II) For $\varphi \in C_c^\infty(\mathbb{C} \times \mathbb{C}^n)$, let $\langle f,\varphi\rangle := -\frac{1}{\pi}\int \frac{\langle g, \tau_{(\xi,0)}\varphi\rangle}{\xi}\, dm(\xi)$, where m is Lebesgue measure on $\mathbb{C}$. (3)

Then $f \in \mathcal{D}'(\mathbb{C} \times \mathbb{C}^n)$

We check this:

By I) $\langle f,\varphi\rangle$ is well-defined. (ξ^{-1} is integrable over $\mathbb{C}$).

Clearly, f is linear. To prove that it is continuous, it suffices to prove that it is continuous on $C^\infty_{K_R \times B}(\mathbb{C} \times \mathbb{C}^n)$, by properties of LF spaces [Trèves [6]], where $K_R = \{|z| \leq R\}$, B compact in $\mathbb{C}^n$. We can also take $K_R \times B$ so big that supp $g \subset K_R \times B$, as in I) above. As in I) supp $(\xi \mapsto \langle g, \tau_{(\xi,0)}\varphi\rangle) \subset K_{2R}$ for supp $\varphi \subset K_R \times B$. Introducing polar coordinates, (r,θ), on $\mathbb{C}$, we get

$$|\langle f,\varphi\rangle| = \frac{1}{\pi}\left|\int_{\mathbb{C}} \frac{\langle g, \tau_{(\xi,0)}\varphi\rangle}{\xi}\, dm(\xi)\right| = \frac{1}{\pi}\left|\int_{K_{2R}} \frac{\langle g, \tau_{(\xi 0)}\varphi\rangle}{\xi}\, dm(\xi)\right|$$

$$= \frac{1}{\pi}\left|\int_0^{2\pi}\int_0^{2R} \langle g, \tau_{(re^{i\theta},0)}\varphi\rangle \frac{dr\, d\theta}{e^{i\theta}}\right|$$

$$\leq \frac{1}{\pi}\int_0^{2\pi}\int_0^{2R} |\langle g, \tau_{(re^{i\theta},0)}\varphi\rangle|\, d\, r\, d\, \theta \leq 4R^2 |\langle g, \tau_{(\xi_0,0)}\varphi\rangle|,$$ where

$|\langle g, \tau_{(\xi,0)}\rangle|$ attains its maximum on $\{|\xi| \leq 2R\} = K_{2R}$ at $\xi_0 \in K_{2R}$.

For $\xi \in K_{2R}$ we have supp $\tau_{(\xi,0)}\varphi \subset K_{3R} \times B$, and the continuity of g gives for some constant $C > 0$ that $|\langle g, \tau_{(\xi_0,0)}\varphi\rangle|$

$\leq C \sup_D \sup_{(z,w)\in K_{3R}\times B} |D(\tau_{(\xi_0,0)}\varphi)(z,w)|$, where D means differentiation monomial in real coordinates of $\mathbb{C} \times \mathbb{C}^n$, and the D-sup is taken over monomials of order less then some integer. We have $D(\tau_{(\xi_o,0)}\varphi) = \tau_{(\xi_o,0)}(D\varphi)$, and for $(z,w) \in \mathbb{C} \times \mathbb{C}^n$ we have

$$|\tau_{(\xi_o,0)}(D\varphi)(z,w)| = |(D\varphi)(z-\xi_o,w)| \leq \sup_{(z,w)\in K_R\times B} |(D\varphi)(z,w)|,$$

since $\operatorname{supp} \varphi \subset K_R \times B$. Thus $|\langle g, \tau_{(\xi_o,0)}\varphi\rangle| \leq C \sup_D \sup_{(z,w)\in K_R\times B}$

$|(D\varphi)(z,w)|$. By the above we then get $|\langle f,\varphi\rangle| \leq 4R^2 C \sup_D$

$\sup_{(z,w)\in K_R\times B} |(D\varphi)(z,w)|$ for $\varphi \in C^\infty_{K_R\times B}(\mathbb{C}\times\mathbb{C}^n)$, which proves

continuity, since $\sup_D \sup_{(z,w)\in K_R\times B}|(D\varphi)(z,w)|$ is a semi-norm on

$C^\infty_{K_R\times B}(\mathbb{C}\times\mathbb{C}^n)$. (Order of D is bounded).

III) Remember now that in the beginning of the proof we multiplied the original g, call it here g_o, by a $C^\infty_c(\mathbb{C}\times\mathbb{C}^n)$ - Function with support in the given U and being equal to $\underline{1}$ in a nbh. U' of $K \times L$. More accurately we do this as follows: Let U_1 be open in $\mathbb{C}$ and U_2 open in $\mathbb{C}^n$, such that $K\times L \subset U_1 \times U_2 \subset U$, and let $\theta_1 \in C^\infty_c(U_1)$ and $\theta_2 \in C^\infty_c(U_2)$ be such $\theta_1 = 1$ in a nbh. U_1' of K and $\theta_2 = 1$ in a nbh. U'_2 of L. Then let

$\theta \in C^\infty_c(U)$ be $\theta(z,w) := \theta_1(z)\cdot\theta_2(w)$. We have then $\theta = 1$ in $U' := U_1' \times U_2'$.

We take our g as $g := \theta g_o$. If $\varphi \in C_c^\infty(C \times C^n)$ has support in $C \times U_2'$, then we get $\langle \partial g/\partial \overline{w}^k, \varphi\rangle = \langle \theta_1 \frac{\partial \theta_2}{\partial \overline{w}^k} g_o, \varphi\rangle +$ $\langle \theta_1 \theta_2 \frac{\partial g_o}{\partial \overline{w}^k}, \varphi\rangle = 0$, since $\partial g_o/\partial \overline{w}^k = 0$ and since

$\partial \theta_2/\partial \overline{w}^k = 0$ in U_2'.

With g constructed in this way, f is the distribution we seek. (4)

Check: Let $\varphi \in C_c^\infty(U')$. Then for any $\xi \in C$ we have supp $\tau_{(\xi,0)}\varphi \subset C \times U_2'$, and the above gives: $\langle \partial f/\partial \overline{w}^k, \varphi\rangle =$

$$- \langle f, \partial\varphi/\partial \overline{w}^k\rangle$$

$$= \frac{1}{\pi}\int_C \frac{\langle g, \tau_{(\xi,0)} \frac{\partial\varphi}{\partial \overline{w}^k}\rangle}{\xi} dm(\xi) = \frac{1}{\pi}\int_C \frac{\langle g, \frac{\partial}{\partial \overline{w}^k}(\tau_{(\xi,0)}\varphi)\rangle}{\xi} dm(\xi)$$

$$= -\frac{1}{\pi}\int_C \frac{\langle \partial g/\partial \overline{w}^k, \tau_{(\xi,0)}\varphi\rangle}{\xi} dm(\xi) = -\frac{1}{\pi}\int_C \frac{0}{\xi} dm(\xi) = 0 .$$

Thus $\frac{\partial f}{\partial \overline{w}^k} = 0$ in U', which is part of what we need. Further,

for $\varphi \in C_c^\infty(U')$, we have $\langle \partial f/\partial \overline{z}, \varphi\rangle = - \langle f, \partial\varphi/\partial \overline{z}\rangle$

$$= \frac{1}{\pi}\int_C \frac{\langle g, \tau_{(\xi,0)} \frac{\partial\varphi}{\partial \overline{z}}\rangle}{\xi} dm(\xi) = \frac{1}{\pi}\int_{C-\{0\}} \frac{\langle g, \tau_{(\xi,0)} \frac{\partial\varphi}{\partial \overline{z}}\rangle}{\xi} dm(\xi)$$

$$= -\frac{1}{\pi}\int_{C-\{0\}} \frac{1}{\xi} \frac{1}{\partial\overline{\xi}} \langle g, \tau_{(\xi,0)}\varphi\rangle dm(\xi),$$ by I). Since $d\overline{\xi}\wedge d\xi = 2\,i\,dm(\xi)$,

we have $$\langle \partial f/\partial \overline{z}, \varphi\rangle = -\frac{1}{2\pi i}\int_{C-\{0\}} \frac{1}{\xi}\frac{\partial}{\partial\overline{\xi}} \langle g, \tau_{(\xi,0)}\varphi\rangle d\overline{\xi}\wedge d\xi$$

$$= -\frac{1}{2\pi i} \int_{C-\{0\}} \frac{1}{\xi} d\langle g, \tau_{(\xi,0)}\varphi\rangle \wedge d\xi \ .$$

Now if $\alpha \in C^\infty(C)$, then in $C - \{0\}$ we have $d(\alpha\xi^{-1}d\xi)$ $= d(\alpha\xi^{-1}) \wedge d\xi = \xi^{-1}d\alpha \wedge d\xi - \alpha\xi^{-2}d\xi\wedge d\xi = \xi^{-1}d\alpha\wedge d\xi$. In the above calculation this gives $\langle \partial f/\partial\bar z, \varphi\rangle = -\frac{1}{2\pi i} \int_{C-\{0\}} d(\langle g, \tau_{(\xi,0)}\varphi\rangle \frac{1}{\xi} d\xi)$

$$= \lim_{\epsilon\to 0} -\frac{1}{2\pi i} \int_{2\pi|\xi|\geq\epsilon} d(\langle g,\tau_{(\xi,0)}\varphi\rangle \tfrac{1}{\xi}d\xi) = \lim_{\epsilon\to 0} \frac{1}{2\pi i} \int_{|\xi|=\epsilon} \langle g,\tau_{(\xi,0)}\varphi\rangle \tfrac{1}{\xi}d\xi$$

$= \lim_{\epsilon\to 0} \frac{1}{2\pi} \int_0 \langle g, \tau_{(\epsilon e^{i\theta},0)}\varphi\rangle d\theta = \langle g, \tau_{(0,0)}\varphi\rangle = \langle g,\varphi\rangle$, since $\xi \to \langle g, \tau_{(\xi,0)}\varphi\rangle$ is continuous. We see now that $\partial f/\partial\bar z = g$ in U'. Since this g equals the original g, called g_o in part III), in U', the lemma is proved.

We now need some notation :

For U open in C^n, let $D'^{(p,q)}(U)$ be the forms of type (p,q) with distributional coefficients ("currents"). Thus $\omega \in D'^{(p,q)}(U)$ can be written $\omega = \sum_{I,J} a_{IJ}\, dz^I \wedge d\bar z^J$, with $a_{IJ} \in D'(U)$, I and J multi-indices of length p and q respectively, and dz^I = $dz^{i_1} \wedge \ldots \wedge dz^{i_p}$ if $I = (i_1,\ldots,i_p)$ etc.

The operator $\bar\partial$ acts as usual: $\bar\partial\omega = \sum_{IJj} \frac{\partial a_{IJ}}{\partial \bar z^j} d\bar z^j \wedge dz^I \wedge d\bar z^J$.

We now have Grothencieck's lemma, the proof of which is found in Narasimhan [4]. (In the C^∞- case) Since it is short, we give it for completeness.

<u>Grothendieck's lemma</u>

Let $K_1,\ldots,K_n$ be compact sets in C and $S = K_1 \times \ldots \times K_n \subset C^n$. Let $\omega \in \mathcal{D}'^{(p,q)}(U)$, where $q \geq 1$ and U is a nbh. of S. If $\bar\partial\omega = 0$, then there exists $\omega' \in \mathcal{D}'^{(p,q-1)}(U)$ such that $\bar\partial\omega' = \omega$ in a nbh. of S.

Proof: If $1 \leq \nu \in \mathbb{Z}$, let $A_\nu^{p,q} = A_\nu^{p,q}(S)$ be the space of elements $\omega \in \mathcal{D}'^{(p,q)}(U)$ defined in a nbh. $U = U(\omega)$ of S and such that ω does not involve $d\bar{z}^\nu, \ldots, d\bar{z}^n$. Thus $\omega = \sum_{IJ} a_{IJ}\, dz^I \wedge d\bar{z}^J$ where $J = (j_1, \ldots, j_q)$ with $1 \leq j_1 < \ldots < j_q \leq \nu-1$. If $\nu > n$, then $A_\nu^{p,q}$ is the space of elements $\omega \in \mathcal{D}'^{(p,q)}(U)$ defined in a nbh. $U = U(\omega)$ of S. Also if $\omega \in A_1^{p,q}$ and $q \geq 1$, then clearly $\omega = 0$, and thus the lemma is trivial if $\omega \in A_1^{p,q}$. Suppose the lemma is true for all $\tilde{\omega} \in A_\nu^{p,q}$, and let $\omega \in A_{\nu+1}^{p,q}$. We can write $\omega = d\bar{z}^\nu \wedge \omega_1 + \omega_2$, with $\omega_1 \in A_\nu^{p,q-1}$ and $\omega_2 \in A_\nu^{p,q}$. If $\bar{\partial}\omega = 0$, then $-d\bar{z}^\nu \wedge \bar{\partial}\omega_1 + \bar{\partial}\omega_2 = 0$. Since ω_1 and ω_2 do not involve $d\bar{z}^\nu, \ldots, d\bar{z}^n$, we have $\partial\omega_1/\partial\bar{z}^j = 0$ and $\partial\omega_2/\partial\bar{z}^j = 0$, (componentwise differentiation), for $j = \nu+1, \ldots, n$. By our first lemma there exists $\chi' \in \mathcal{D}'^{(p,q-1)}(U)$ in a nbh. U' of S, with $\partial\chi'/\partial\bar{z}^j = 0$ for $j = \nu+1, \ldots, n$ and $\partial\chi'/\partial\bar{z}^\nu = \omega_1$. Multiplying χ' by a $C_c^\infty(U')$-function which is equal to 1 in a nbh. of S, we see that there exists $\chi \in \mathcal{D}'^{(p,q-1)}(C^n)$ with $\partial\chi/\partial\bar{z}^j = 0$ for $j = \nu+1, \ldots, n$ and $\partial\chi/\partial\bar{z}^\nu = \omega_1$ in a nbh. of S. This implies that $\omega - \bar{\partial}\chi \in A_\nu^{p,q}$. Since $\bar{\partial}(\omega - \bar{\partial}\chi) = \bar{\partial}\omega = 0$, there is, by the induction hypothesis, an element $\psi \in \mathcal{D}'^{(p,q-1)}(C^n)$ with $\omega - \bar{\partial}\chi = \bar{\partial}\psi$ in a nbh. of S.

We further need the following theorem, a proof of which in the case of a Riemann surface can be found in Gunning [2]. To generalize that proof to the case of arbitrary dimension, we need a Cauchy formula in several variables. Since we will use differential forms, Stoke's theorem etc., it is convenient to use the Cauchy-Martinelli formula. This reads as follows: If f is a holomorphic function in a nbh. of $\xi + K_R \subset C^n$, where $K_R = \{z \in C^n \mid |z| \leq R\}$, then

$$f(\xi) = (-1)^{\frac{n(n+1)}{2}} \cdot \frac{(n-1)!}{(2\pi i)^n} \int_{\xi+S_R} \frac{f(z)}{|z-\xi|^{2n}}\, \omega(z-\xi)\,, \quad \text{where} \quad S_R = \{|z| = R\}$$

and $\omega(z) := \sum_{k=1}^{n} (-1)^k \bar{z}^k\, dz^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^1 \wedge \ldots \wedge \widehat{d\bar{z}^k} \wedge \ldots \wedge d\bar{z}^n$. ($\wedge$ means "omission", as usual) There is a simple proof in Lang [3].

The generalization of the theorem in Gunning is:

Let $T \in \mathcal{D}'(U)$, where U is open in C^n, and assume that $\partial T/\partial \bar{z}^j = 0$ for $j = 1,\ldots,n$. Then T is a holomorphic function in U .

Proof: Rewriting the Cauchy-Martinelli formula, interchanging z and ξ, and putting $A := (-1)^{\frac{n(n+1)}{2}} \frac{(n-1)!}{(2\pi i)^n}$, we have $f(z) = A \int_{S_R} f(z+\xi)\frac{\omega(\xi)}{|\xi|^{2n}}$ for f holomorphic near $z + K_R$. In particular, for $f \equiv 1$ we get $1 = A \int_{S_R} \frac{\omega(\xi)}{|\xi|^{2n}}$. If f is only C^∞ in a nbh. of z, we get then:

$$\left|f(z) - A \int_{S_R} f(z+\xi)\frac{\omega(\xi)}{|\xi|^{2n}}\right| = \left|A \int_{S_R} f(z)\frac{\omega(\xi)}{|\xi|^{2n}} - A \int_{S_R} f(z+\xi)\frac{\omega(\xi)}{|\xi|^{2n}}\right|$$

$= \left|A \int_{S_R} (f(z) - f(z+\xi))\frac{\omega(\xi)}{|\xi|^{2n}}\right|$. This quantity can be made arbitrarily small by taking R small enough, since f is continuous. Thus we see

$$f(z) = \lim_{R\to 0} A \int_{S_R} f(z+\xi)\frac{\omega(\xi)}{|\xi|^{2n}} \quad \text{for } f \ C^\infty \text{ near } z\,. \tag{5}$$

To prove that T is holomorphic it is sufficient to prove that it is C^∞, and to show that, we will write any $\varphi \in C_c^\infty(U)$ in a special form so that we can use the conditions given on T . Let then, for $\epsilon > 0$, $U_\epsilon := \{z \in C^n \mid \operatorname{dist}(z, C^n - U) > \epsilon\}$, and let $\varphi \in C_c^\infty(U_\epsilon) \subset C_c^\infty(U)$. Further let $\rho \in C_c^\infty(C^n)$ be such that $\rho(\xi) = 1$ for

$|\xi| < \epsilon/2$, and $\operatorname{supp} \rho \subset \{|\xi| < \epsilon\}$. By (5) we get, for $z \in U_\epsilon$,

$\varphi(z) = \lim\limits_{R\to 0} A \int\limits_{S_R} \varphi(z+\xi)\dfrac{\rho(\xi)}{|\xi|^{2n}}\omega(\xi)$, since $\rho = 1$ on S_R for $R < \epsilon/2$.

By Stoke we get, since the orientation of S_R is outward, observing that $\xi \to f(z+\xi)\rho(\xi)$ is defined for all $\xi \in C$ since

$$z \in U_\epsilon : \ \varphi(z) = -\lim_{R\to 0} A \int\limits_{|\xi|\geq R} d[\varphi(z+\xi)\frac{\rho(\xi)}{|\xi|^{2n}}\,\omega(\xi)] \tag{6}$$

Here the d is w.r.t. ξ. We have $d[\varphi(z+\xi)\dfrac{\rho(\xi)}{|\xi|^{2n}}\,\omega(\xi)] =$

$d\varphi(z+\xi)\wedge\dfrac{\rho(\xi)}{|\xi|^{2n}}\,\omega(\xi) + \varphi(z+\xi)d[\dfrac{\rho(\xi)}{|\xi|^{2n}}\,\omega(\xi)]$. Since each term of ω

contains $d\xi'\wedge\ldots\wedge d\xi^n$, we see $d\varphi(z+\xi)\wedge\dfrac{\rho(\xi)}{|\xi|^{2n}}\,\omega(\xi) = \bar\partial\varphi(z+\xi)$

$$\wedge\frac{\rho(\xi)}{|\xi|^{2n}}\,\omega(\xi) = \sum_{j=1}^{n} \frac{\partial\varphi(z+\xi)}{\partial\bar\xi^j}\,d\bar\xi^j\wedge\frac{\rho(\xi)}{|\xi|^{2n}}\,\omega(\xi)$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial\varphi}{\partial\bar z^j}(z+\xi)\cdot\frac{\rho(\xi)}{|\xi|^{2n}}\cdot(-1)^k\,\bar\xi^k\,d\bar\xi^j\wedge d\xi'\wedge\ldots\wedge d\xi^n\wedge d\bar\xi'\wedge\ldots\wedge\widehat{d\bar\xi^k}\wedge\ldots\wedge d\bar\xi^n$$

$$= \sum_{j=1}^{n}\frac{\partial\varphi}{\partial\bar z^j}(z+\xi)\frac{\rho(\xi)}{|\xi|^{2n}}\,\bar\xi^j\cdot(-1)^{n-1}d\xi'\wedge\ldots\wedge d\xi^n\wedge d\bar\xi'\wedge\ldots\wedge d\bar\xi^n$$

$$= \sum_{j=1}^{n}\frac{\partial\varphi}{\partial\bar z^j}(z+\xi)\frac{\rho(\xi)}{|\xi|^{2n}}\,\bar\xi^j\cdot(-1)^{n-1+\frac{n(n+1)}{2}}\,d\bar\xi'\wedge d\xi'\wedge\ldots\wedge d\bar\xi^n\wedge d\xi^n$$

$= (-1)^{n-1+\frac{n(n+1)}{2}}\cdot(2i)^n\cdot\sum\limits_{j=1}^{n}\dfrac{\partial\varphi}{\partial\bar z^j}(z+\xi)\cdot\dfrac{\rho(\xi)}{|\xi|^{2n}}\cdot\bar\xi^j\,d\,m(\xi)$, where dm is Lebesgue measure on C^n.

Further $d[\dfrac{\rho(\xi)}{|\xi|^{2n}}\,\omega(\xi)] = d[\dfrac{\rho(\xi)}{|\xi|^{2n}}]\wedge\omega(\xi) + \dfrac{\rho(\xi)}{|\xi|^{2n}}d\,\omega(\xi)$

$$= \bar\partial[\frac{\rho(\xi)}{|\xi|^{2n}}]\wedge\omega(\xi) + \frac{\rho(\xi)}{|\xi|^{2n}}d\,\omega(\xi)$$

$$= (-1)^{n-1+\frac{n(n+1)}{2}}(2i)^n\sum_{j=1}^{n}\frac{\partial}{\partial\bar\xi^j}[\frac{\rho(\xi)}{|\xi|^{2n}}]\cdot\bar\xi^j dm(\xi) + \frac{\rho(\xi)}{|\xi|^{2n}}\sum_{j=1}^{n}(-1)^j d\bar\xi^j\wedge d\xi'\wedge\ldots$$
$$\wedge d\xi^n\wedge d\bar\xi'\wedge\ldots\wedge\widehat{d\bar\xi^j}\wedge\ldots\wedge d\bar\xi^n$$

$$= (-1)^{n-1+\frac{n(n+1)}{2}}(2i)^n \sum_{j=1}^{n} \frac{\partial}{\partial\bar{\xi}^j}\left[\frac{\rho(\xi)}{|\xi|^{2n}}\right]\cdot\bar{\xi}^j dm(\xi) + (-1)^{n-1}\cdot\frac{\rho(\xi)}{|\xi|^{2n}} \sum_{j=1}^{n} d\xi'\wedge\ldots$$

$$\ldots\wedge d\xi^n\wedge d\bar{\xi}'\wedge\ldots\wedge d\bar{\xi}^n$$

$$= (-1)^{n-1+\frac{n(n+1)}{2}}(2i)^n\left\{n\frac{\rho(\xi)}{|\xi|^{2n}} + \sum_{j=1}^{n} \frac{\partial}{\partial\bar{\xi}^j}\left[\frac{\rho(\xi)}{|\xi|^{2n}}\right]\cdot\bar{\xi}^j\right\}dm(\xi)$$

$$= (-1)^{n-1+\frac{n(n+1)}{2}}(2i)^n\tilde{h}(\xi)\,dm(\xi),$$

where we have put $\tilde{h}(\xi) := n\frac{\rho(\xi)}{|\xi|^{2n}} + \sum_{j=1}^{n} \frac{\partial}{\partial\bar{\xi}^j}\left[\frac{\rho(\xi)}{|\xi|^{2n}}\right]\bar{\xi}^j$. Putting all

this together, we get $d\left[\varphi(z+\xi)\frac{\rho(\xi)}{|\xi|^{2n}}\,\omega(\xi)\right]$

$$= (-1)^{n-1+\frac{n(n+1)}{2}}(2i)^n\left\{\sum_{j=1}^{n} \frac{\partial\varphi}{\partial\bar{z}^j}(z+\xi)\frac{\rho(\xi)}{|\xi|^{2n}}\bar{\xi}^j + \varphi(z+\xi)\tilde{h}(\xi)\right\}dm(\xi).$$

In (6) above we now get, since $A = (-1)^{\frac{n(n+1)}{2}}\frac{(n-1)!}{(2\pi i)^n}$:

$$\varphi(z) = \lim_{R\to 0}(-1)^{n+\frac{n(n+1)}{2}}(2i)^n\cdot(-1)^{\frac{n(n+1)}{2}}\frac{(n-1)!}{(2\pi i)^n}\int_{|\xi|\geq R}$$

$$\left\{\sum_{j=1}^{n} \frac{\partial\varphi}{\partial\bar{z}^j}(z+\xi)\frac{\rho(\xi)}{|\xi|^{2n}}\bar{\xi}^j+\varphi(z+\xi)\tilde{h}(\xi)\right\}dm(\xi)$$

$$= \lim_{R\to 0}\left(-\frac{1}{\pi}\right)^n(n-1)!\left\{\sum_{j=1}^{n}\int_{|\xi|\geq R}\frac{\partial\varphi}{\partial\bar{z}^j}(z+\xi)\frac{\rho(\xi)}{|\xi|^{2n}}\bar{\xi}^j dm(\xi)+\int_{|\xi|\geq R}\varphi(z+\xi)\tilde{h}(\xi)dm(\xi)\right\}.$$

Since the n first integrals all converge as $R\to 0$, then so does the last, and we get

$$\varphi(z) = \left(-\frac{1}{\pi}\right)^n(n-1)!\left\{\sum_{j=1}^{n}\int_{C^n}\frac{\partial\varphi}{\partial\bar{z}^j}(z+\xi)\frac{\rho(\xi)}{|\xi|^{2n}}\bar{\xi}^j dm(\xi)+\int_{C^n}\varphi(z+\xi)\tilde{h}(\xi)dm(\xi)\right\}.$$

Let $h_j(z) := \left(-\frac{1}{\pi}\right)^n(n-1)!\int_{C^n}\varphi(z+\xi)\frac{\rho(\xi)}{|\xi|^{2n}}\bar{\xi}^j dm(\xi)$, and

$$h(\xi) := \left(-\frac{1}{\pi}\right)^n(n-1)!\tilde{h}(\xi)$$

$$= \left(-\frac{1}{\pi}\right)^n(n-1)!\sum_{j=1}^{n}\left\{\frac{\rho(\xi)}{|\xi|^{2n}}+\frac{\partial}{\partial\bar{\xi}^j}\left[\frac{\rho(\xi)}{|\xi|^{2n}}\right]\cdot\bar{\xi}^j\right\} = \left(-\frac{1}{\pi}\right)^n(n-1)!\sum_{j=1}^{n}\frac{\partial}{\partial\bar{\xi}^j}\left[\rho(\xi)\frac{\bar{\xi}^j}{|\xi|^{2n}}\right].$$

We then get, (supp ρ is compact):

$$\varphi(z) = \sum_{j=1}^{n} \frac{\partial h_j}{\partial \bar{z}^j}(z) + \int_{C^n} \varphi(z+\xi)h(\xi)dm(\xi) .$$

$$\text{Here } h_1,\ldots,h_j \in C_c^\infty(U), \text{ and also } g \in C_c^\infty(U), \tag{7}$$

$$\text{where } g(z) := \int_{C^n} \varphi(z+\xi)h(\xi)dm(\xi)$$

This is clear, since supp $\varphi \subset U_\varepsilon$ and supp $\rho \subset \{|\xi| < \varepsilon\}$.

If we further assume that supp $\varphi \subset U_{2\varepsilon} \subset U_\varepsilon$, then supp $g \subset U_{\varepsilon/2}$ (8)

In fact, let $z \in U - U_\varepsilon$. If $|\xi| \geq \varepsilon$, then $h(\xi) = 0$ since $\rho(\xi) = 0$, and thus $\varphi(z+\xi)h(\xi) = 0$. If $|\xi| < \varepsilon$, then $z + \xi \in U - U_{2\varepsilon}$, and thus $\varphi(z+\xi)h(\xi) = 0\cdot h(\xi) = 0$, which proves (8). Let further $\theta \in C_c^\infty(U)$ be such that $\theta = 1$ in a nbh. of $\overline{U_{\varepsilon/2}}$. Then the function

$$C^n \ni \xi \to \langle \theta T)_z, h(\xi - z)\rangle = \langle \theta T, \tau_\xi \check{h}\rangle \text{ is } C^\infty, \text{ where } \tau_\xi$$
$$\text{is translation, } \check{h}(z) := h(-z) \text{ and } (\theta T)_z \text{ means that} \tag{9}$$
$$\theta T \text{ acts w.r.t. } z .$$

That $\langle \theta T, \tau_\xi \check{h}\rangle$ is well defined, follows since supp θT is compact and the C^∞ statement follows as in part I) of the proof of our first lemma. By (8) we have $\langle T,g\rangle = \langle \theta T,g\rangle$, and further

$$\begin{aligned} \langle T,g\rangle &= \langle \theta T,g\rangle = \langle \theta T, \int_{C^n} \varphi(z+\xi)h(\xi)dm(\xi)\rangle \\ &= \langle \theta T, \int_{C^n} h(\xi - z)\varphi(\xi)dm(\xi)\rangle = \int_{U_{2\varepsilon}} \langle \theta T, \tau_\xi \check{h}\rangle \varphi(\xi)dm(\xi) \end{aligned} \tag{10}$$

We must prove the last equality, and after that we will quickly finish the proof of the theorem. Consider $\int_{C^n} h(\xi - z)\varphi(\xi)dm(\xi)$ as a limit of Riemann sums of the form $\sum_\alpha h(\xi_\alpha - z)\varphi(\xi_\alpha)m(S_\alpha)$, where $m(S_\alpha)$ is the measure of a rectangle S_α containing ξ_α . More generally, if D is a differentiation monomial of order p in the real components of $z \in \mathbb{C}^n$, then we have $D\int_{\mathbb{C}^n} h(\xi-z)\varphi(\xi)dm(\xi)$ $= \lim \sum_\alpha (-1)^p (Dh)(\xi_\alpha - z)\varphi(\xi_\alpha)m(S_\alpha)$. These sums converge uniformly w.r.t. z on compact sets, and thus $\lim(z \to \sum_\alpha h(\xi_\alpha - z)\varphi(\xi_\alpha)m(S_\alpha))$ $= (z \to \int_{\mathbb{C}^n} h(\xi-z)\varphi(\xi)dm(\xi))$ in the space $C_c^\infty(\mathbb{C}^n)$. By continuity of θT on this space we get $\lim\langle \theta T, (z \to \sum_\alpha h(\xi_\alpha - z)\varphi(\xi_\alpha)m(S_\alpha))\rangle$ $= \langle \theta T, \int_{\mathbb{C}^n} h(\xi-z)\varphi(\xi)dm(\xi)\rangle$. The left hand side of this equals $\lim \sum_\alpha \langle (\theta T)_z, h(\xi_\alpha - z)\rangle\varphi(\xi_\alpha)m(S_\alpha)$. Since this is a Riemann sum for $\int_{\mathbb{C}^n} \langle (\theta T)_z, h(\xi-z)\rangle\varphi(\xi)dm(\xi)$, (by (9) above the integrand is C^∞ with support in $\operatorname{supp}\varphi$), we get $\langle \theta T, \int_{\mathbb{C}^n} h(\xi-z)\varphi(\xi)dm(\xi)\rangle$

$$= \int_{\mathbb{C}^n} \langle (\theta T)_z, h(\xi-z)\rangle\varphi(\xi)dm(\xi) = \int_{\mathbb{C}^n} \langle \theta T, \tau_\xi \check{h}\rangle\varphi(\xi)dm(\xi)$$

$$= \int_{U_{2\varepsilon}} \langle \theta T, \tau_\varepsilon \check{h}\rangle\varphi(\xi)dm(\xi),$$ for $\varphi \in C_c^\infty(U_{2\varepsilon})$. Thus (10) above is proved.

By (7) and (10) above we get for $\varphi \in C_c^\infty(U_{2\varepsilon})$, using the fact that $\partial T/\partial\bar{z}^j = 0$ for $j = 1,\ldots,n$:

$$\langle T,\varphi\rangle = \langle T, \sum_{j=1}^n \partial h_j/\partial\bar{z}^j + \int_{\mathbb{C}^n} \varphi(z+\xi)h(\xi)dm(\xi)\rangle$$

$$= \sum_{j=1}^n \langle T, \partial h_j/\partial\bar{z}^j\rangle + \langle T, \int_{\mathbb{C}^n} \varphi(z+\xi)h(\xi)dm(\xi)\rangle$$

$$= -\sum_{j=o}^n \langle \partial T/\partial\bar{z}^j, h_j\rangle + \langle \theta T, \int_{\mathbb{C}^n} \varphi(z+\xi)h(\xi)dm(\xi)\rangle$$

$$= \int_{U_{2\varepsilon}} \langle \theta T, \tau_\xi \check{h}\rangle\varphi(\xi)dm(\xi),$$ and here $\xi \to \langle \theta T, \tau_\xi \check{h}\rangle$,

which is independent of φ, is a C^∞-function, by (9) above. Thus T equals the C^∞-function $\xi \to \langle \theta T, \tau_\xi \check{h}\rangle$ in $U_{2\varepsilon}$. Since this holds for all $\varepsilon > 0$, we have that T is a C^∞-function, and thus holomorphic.

QED.

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